

ON THE PURITY OF MAXIMAL WEAKLY SEPARATED SET FAMILIES

HWANCHUL YOO

ABSTRACT. We present a short proof that every maximal family of weakly separated subsets of $[n]$ of cardinality between $[a, b]$ have the same size. Our proof is direct and only uses elementary combinatorics of lattice paths.

1. INTRODUCTION

Quantized coordinate ring of the flag variety is called the quantum flag variety. Leclerc and Zelevinsky[2] studied this ring, and came up with weakly separated set families.

Definition 1.1. Two subsets $A, B \subset [n]$ are said to be weakly separated if at least one of the following two conditions holds:

- $|A| \geq |B|$ and $B \setminus A$ can be partitioned into a disjoint union $B \setminus A = B' \sqcup B''$ so that $b' < a < b''$ for all $b' \in B'$, $a \in A \setminus B$ and $b'' \in B''$
- $|B| \geq |A|$ and $A \setminus B$ can be partitioned into a disjoint union $A \setminus B = A' \sqcup A''$ so that $a' < b < a''$ for all $a' \in A'$, $b \in B \setminus A$ and $a'' \in A''$

Two quantum Plücker coordinates quasi-commute whenever their indexing sets are weakly separated. It is conjectured that every maximal family of pairwise weakly separated sets have the size $\binom{n+1}{2} + 1$. Scott[4] studied the quantized coordinate ring of Grassmannian, and proposed a similar conjecture that every maximal weakly separated k -set families have the size $k(n - k) + 1$. These conjectures have been referred to as *purity conjectures*.

We prove the purity conjecture for the subsets of $[n]$ whose cardinality is between a and b for some integers $a \leq b$. This proves simultaneously Leclerc-Zelevinsky's and Scott's conjectures. Previously, Danilov-Karzanov-Koshevoy[1] and Oh-Postnikov-Speyer[3] independently proved them using some planar graphic machineries. Our proof is more direct in the sense that we only use one operation on weakly separated set families, namely the truncation.

Remark 1.2. This proof has been found during the discussion with Suho Oh when Oh-Postnikov-Speyer were preparing their paper. The author was reluctant to write it down at the time because Oh-Postnikov-Speyer's method was more general and revealed more structure. Although, since it is always good to have different proofs on hand, we decided to present it now.

Acknowledgement. The author thanks to Suho Oh for introducing him the problem. Also grateful to Alexander Postnikov for encouraging the author to write this down.

2. SUBSETS AND LATTICE PATHS

In this paper, a lattice path will mean a path that connects two lattice points by moving only to the east and north directions. Let P be a lattice path, and denote by P_i the i -th step of P . Namely, P_i is an either horizontal or vertical unit length segment of P . There is a standard way to identify a k -subset of $[n]$ with a lattice path from $(0, 0)$ to $(k, n - k)$. Given a k -subset, the corresponding path P has horizontal P_i if i is in the subset, and vertical P_i otherwise. In this case, we will denote the subset by $[P]$.

Now, let us define a condition between two lattice paths that will play a role of weak separation.

Definition 2.1. Let P, Q be a pair of lattice paths. We say that they *conflict at i and j* ($i < j$) if they have the configuration as in Figure 1. In this case, we say that P *conflicts with Q from above*, and Q *conflicts with P from below*. Note that this is equivalent to satisfying following conditions:

- (1) $i \in [P], j \notin [P], i \notin [Q], j \in [Q]$,
- (2) the number of elements of $[P]$ smaller than i is less than the number of elements of $[Q]$ smaller than i ,
- (3) the number of elements of $[Q]$ greater than j is less than the number of elements of $[P]$ greater than j .

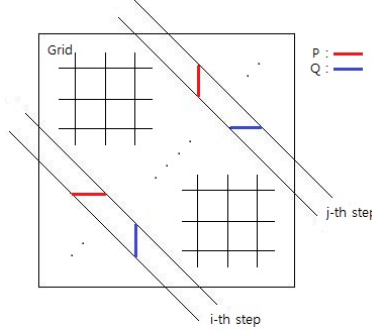


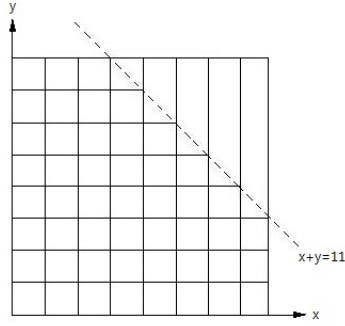
FIGURE 1. P and Q conflict at i and j

Lemma 2.2. Two k -subsets $[P]$ and $[Q]$ are weakly separated if and only if the lattice paths P and Q don't conflict anywhere.

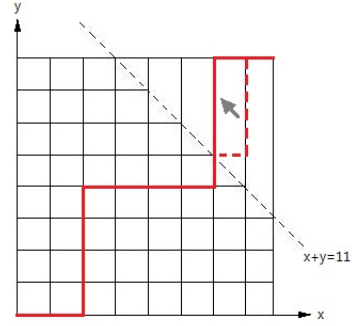
Proof. Let P conflict with Q at i and j ($i < j$) from above. From the equivalent description of the definition, we have $i \in [P], j \notin [P], i \notin [Q], j \in [Q]$ and the number of elements of $[P]$ smaller than i (resp. elements of $[Q]$ greater than j) is less than the number of elements of $[Q]$ smaller than i (resp. elements of $[P]$ greater than j). Therefore there exist $s < i$ and $t > j$ such that $s \notin [P], t \in [P], s \in [Q], t \notin [Q]$. So $s < i < j < t$ implies that $[P]$ and $[Q]$ are not weakly separated.

Conversely, let us assume that $[P]$ and $[Q]$ are not weakly separated. Without loss of generality, assume further that there is $s \in [Q], s \notin [P]$ which is the first number contained in only one of $[P]$ and $[Q]$. Let t be the last number contained in only one of $[P]$ and $[Q]$. When $t \in [P]$, let i be the first number contained only in $[P]$, and j be the last number contained only in $[Q]$. Since $[P]$ and $[Q]$ are not weakly separated, j is greater than i , hence $s < i < j < t$. It is clear that P conflicts with Q at i and j from above. When $t \in [Q]$, let i and i' be the first and last numbers contained only in $[P]$. Since $[P]$ and $[Q]$ are not weakly separated, there is j between i and i' contained only in $[Q]$. Either the number of elements of $[Q]$ smaller than j is less than the number of elements of $[P]$ smaller than j , or the number of elements of $[Q]$ greater than j is less than the number of elements of $[P]$ greater than j . Hence P and Q conflict either at $i < j$ or at $j < i'$. \square

Let $a \leq b \leq n$ be nonnegative integers. Define $L_{a,b}^n$ as a partial grid constructed from $(n-a) \times b$ grid by erasing horizontal edges above the hyperplane $x+y=n$, except the ones on the boundary. See Figure 2(a) for an example. We can identify a lattice path P from $(0,0)$ to $(b,n-a)$ contained in $L_{a,b}^n$ with a subset of $[n]$ of cardinality between a and b . To do that, we just delete the elements of $[P]$ greater than n . We will abuse our notation to denote the resulting subset of $[n]$ by $[P]$. It is clear that the correspondence is bijective. Note that this construction is essentially the same as in [3, Section 12] using padding. By Lemma 2.2, it is also clear that $P, Q \subset L_{a,b}^n$ are conflict-free if and only if $[P], [Q] \subset [n]$ are weakly separated.



(a) Partial grid $L_{3,8}^{11}$ with 54 chambers



(b) Truncation from $W_{4,8}^{12}$ to $W_{3,8}^{11}$

FIGURE 2.

As a planar graph, a partial grid $L_{a,b}^n$ consists of rectangular chambers. Let us denote the number of chambers by $|L_{a,b}^n|$. Now we can state our main theorem.

Theorem 2.3. *The size of any maximal conflict-free family of lattice paths in $L_{a,b}^n$ is $|L_{a,b}^n| + 1$.*

Corollary 2.4. *Every maximal family of weakly separated subsets of $[n]$ of cardinality between a and b has the size $|L_{a,b}^n| + 1$. In particular, when $a = b = k$, they have the size $k(n - k) + 1$, and when $a = 0$, $b = n$, they have the size $\binom{n+1}{2} + 1$.*

3. TRUNCATION MAP

In this section, we will prove Theorem 2.3 by using the properties of truncation map.

Let $\mathcal{W}_{a,b}^n$ be the collection of all conflict-free families of lattice paths in $L_{a,b}^n$. Equivalently, it is the collection of all weakly separated families of subsets in $[n]$ of the cardinality between a and b . The truncation map is a natural map

$$\mathcal{T} : \mathcal{W}_{a,b}^n \rightarrow \mathcal{W}_{a-1,b}^{n-1}$$

that corresponds to deleting n from every subset $[P]$ in the family. For $F \in \mathcal{W}_{a,b}^n$, let us abuse our notation to denote the path in $\mathcal{T}(F)$ truncated from a path $P \in F$ by $\mathcal{T}(P)$. See Figure 2(b).

Lemma 3.1. *The map \mathcal{T} is well-defined. Moreover, it preserves the maximality of a family.*

Proof. Let $F \in \mathcal{W}_{a,b}^n$ and $P \in F$. Then the possible difference between P and $\mathcal{T}(P)$ is only at the n -th step. The map substitutes the horizontal n -th step, if there is any, to a vertical one, if possible. It is clear that any two paths $\mathcal{T}(P)$ and $\mathcal{T}(Q)$ don't conflict if P and Q don't. Therefore $\mathcal{T}(F)$ is conflict-free.

For a maximal $F \in \mathcal{W}_{a,b}^n$, assume that $\mathcal{T}(F)$ is not maximal. Then there is a lattice path $P' \notin \mathcal{T}(F)$ contained in $L_{a-1,b}^{n-1}$ that does not conflict with any element of $\mathcal{T}(F)$. Also there is $P \in F$ that conflicts with P' . In this case, they should conflict at n , and P' conflicts with P from above. Now let $\mathcal{T}^{-1}(P')$ be the lattice path in $L_{a,b}^n$ whose each step is the same as P' except at the n -th step. $P' \notin \mathcal{T}(F)$ implies $\mathcal{T}^{-1}(P') \notin F$. Hence there is $Q \in F$ that conflicts with $\mathcal{T}^{-1}(P')$. Similarly as before, $\mathcal{T}^{-1}(P')$ conflicts with Q at n from below in this case. Since P' and $\mathcal{T}^{-1}(P')$ are exactly the same below n -th step, P and Q conflict each other. This is a contradiction. \square

The maximality of a family behaves well with respect to the truncation. The next lemma tells us that we can precisely enumerate the change when we truncate.

Lemma 3.2. $|F| - |\mathcal{T}(F)| = |L_{a,b}^n| - |L_{a-1,b}^{n-1}|$ for all maximal $F \in \mathcal{W}_{a,b}^n$.

Proof. First, let us see why $|F| - |\mathcal{T}(F)| \geq |L_{a,b}^n| - |L_{a-1,b}^{n-1}|$. Let c be a lattice point $(z, n - z - 1)$ in $L_{a,b}^n$ for some $a \leq z < b$. Denote by $\mathcal{T}(F)_c$ the subfamily of paths in $\mathcal{T}(F)$ which pass through c . If there are two distinct paths $P, Q \in \mathcal{T}(F)_c$ such that all of $P, Q, \mathcal{T}^{-1}(P)$ and $\mathcal{T}^{-1}(Q)$ are in F , then at least one pair of $(\mathcal{T}^{-1}(P), Q)$ and $(P, \mathcal{T}^{-1}(Q))$ conflict each other. Therefore for each c , there is at most one path $P \in \mathcal{T}(F)_c$ such that both P and $\mathcal{T}^{-1}(P)$ are in F . The number of possible choices of c is exactly $|L_{a,b}^n| - |L_{a-1,b}^{n-1}|$, hence we are done.

Next, for each c as above, we will actually find such $P \in \mathcal{T}(F)_c$ in the previous paragraph. Let $P \in F$ be the path that satisfies the following conditions:

- (1) P passes through c ,
- (2) n -th step is vertical,
- (3) $[P]$ is lexicographically maximal among those satisfying (1) and (2), when read from n to 1 under the reversed order $n \prec \dots \prec 1$.

Similarly, choose $Q \in F$ to be the path by changing in the above condition (2) to be the horizontal step, and in condition (3) to be lexicographically minimal. Regarding P as in $\mathcal{T}(F)$, we claim that $Q = \mathcal{T}^{-1}(P)$. Assume it is not. Let m be the last step before n where P and Q are different, i.e. $P_i = Q_i$ for all $m < i < n$. It is clear that P_m is vertical and Q_m is horizontal.

Let S be the connected segment of P from $(m+1)$ -th step to the end. Similarly, let T be the connected segment of Q from $(m+1)$ -th step to the end. $Q_m \rightarrow S$ is a lattice path, although not connected from $(0,0)$, that we get by attaching one more step Q_m in front of S . If $Q_m \rightarrow S$ conflicts with some $R \in F$, then it should conflict at m and n from above. Similarly, if $P_m \rightarrow T$ conflicts with some $R' \in F$, then it conflicts at m and n from below. This cannot happen together because R and R' should not conflict each other. Without loss of generality, let $Q_m \rightarrow S$ does not conflict with F . Denote it by S^1 .

For any $1 \leq i < m$, we will show that we can attach one step in front of S^i to get another partial lattice path S^{i+1} that does not conflict with F . It will finish the proof since F is maximal and newly constructed path would not be in it. If we cannot attach such a step, then S^{i+1} conflicts with some $R \in F$ at $m-i$ and j from above, and with some $R' \in F$ at $m-i$ and k from below, for some $j, k > m-i$. Since R and R' does not conflict each other, $R_j = R'_j$ and $R_k = R'_k$. But this implies S^i conflicts with either R or R' which is a contradiction. \square

Using Lemma 3.1, 3.2, we can prove Theorem 2.3.

Proof of Theorem 2.3. First, notice that every maximal $F \in \mathcal{W}_{a,b}^b$ comes from a maximal $F' \in \mathcal{W}_{a,b-1}^b$, vice versa, by including, or excluding, one special lattice path that follows the bottom and right-most boundaries of $L_{a,b}^b$. Using induction on b , this implies that $|F| = |L_{a,b-1}^b| + 2 = |L_{a,b}^b| + 1$. Induction base $b = 0$ case is trivial. Now for any n , we iterate the truncation map to the case $n = b$. Lemma 3.2 finishes the proof. \square

4. EXAMPLE

Let us finish with an example. If we abuse our notation to denote the map $\mathcal{W}_{a,b}^b \rightarrow \mathcal{W}_{a,b-1}^b$ in the above proof by \mathcal{T} , then we can iterate \mathcal{T} to reduce $\mathcal{W}_{a,b}^n$ to $\mathcal{W}_{0,0}^0$. Figure 3 shows how it goes. Let us choose a maximal $F \in \mathcal{W}_{3,3}^6$. Notice that $\mathcal{W}_{3,3}^6$ is equivalent to $\mathcal{W}_{2,3}^5$.

$$\begin{aligned} F &= \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 6\}, \{5, 6, 1\}, \{6, 1, 2\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 4, 6\}, \{3, 4, 6\}\} \\ \mathcal{T}(F) &= \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5\}, \{5, 1\}, \{1, 2\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 4\}, \{3, 4\}\} \\ \mathcal{T}^2(F) &= \{\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4\}, \{4\}, \{1\}, \{1, 2\}, \{1, 3, 4\}, \{1, 2, 4\}, \{1, 4\}\} \end{aligned}$$

$$\begin{aligned}
\mathcal{T}^3(F) &= \{\{1, 2, 3\}, \{2, 3\}, \{3\}, \{\}, \{1\}, \{1, 2\}, \{1, 3\}\} \\
\mathcal{T}^4(F) &= \{\{2, 3\}, \{3\}, \{\}, \{1\}, \{1, 2\}, \{1, 3\}\} \\
\mathcal{T}^5(F) &= \{\{2\}, \{\}, \{1\}, \{1, 2\}\} \\
\mathcal{T}^6(F) &= \{\{2\}, \{\}, \{1\}\} \\
\mathcal{T}^7(F) &= \{\{\}, \{1\}\} \\
\mathcal{T}^8(F) &= \{\{\}\}
\end{aligned}$$

We can check that each of the above families is a maximal weakly separated set family in a intermediate step.

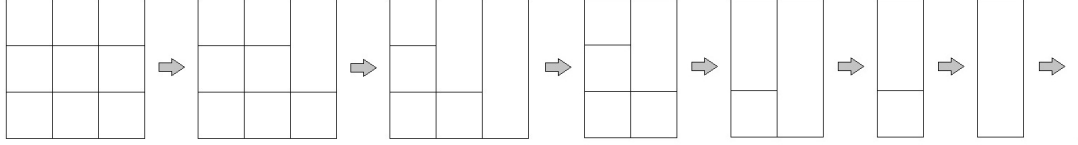


FIGURE 3. $\mathcal{W}_{3,3}^6 = \mathcal{W}_{2,3}^5 \rightarrow \mathcal{W}_{1,3}^4 \rightarrow \mathcal{W}_{0,3}^3 \rightarrow \mathcal{W}_{0,2}^3 \rightarrow \mathcal{W}_{0,2}^2 \rightarrow \mathcal{W}_{0,1}^2 \rightarrow \mathcal{W}_{0,1}^1 \rightarrow \mathcal{W}_{0,0}^0$

REFERENCES

- [1] Vladimir I. Danilov, Alexander V. Karzanov, and Gleb A. Koshevoy. On maximal weakly separated set-systems. *J. Algebraic Comb.*, 32:497–531, December 2010.
- [2] Bernard Leclerc and Andrei Zelevinsky. Quasicommuting families of quantum plucker coordinates. In *Advances in Math. Sciences (Kirillov’s seminar)*, *AMS Translations* 181, pages 85–108, 1998.
- [3] S. Oh, A. Postnikov, and D. E Speyer. Weak Separation and Plabic Graphs. *ArXiv e-prints*, September 2011.
- [4] Josh Scott. Quasi-commuting families of quantum minors. *Journal of Algebra*, 290(1):204–220, Aug.2005.

1530 KIAS 85 HOEGI-RO DONDAEMUN-GU, SEOUL, REPUBLIC OF KOREA
E-mail address: hcyoo@kias.re.kr